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RSS-BASED SENSOR LOCALIZATION WITH UNKNOWN TRANSMIT POWER

Reza M. Vaghefi, Mohammad Reza Gholami, and Erik G. Ström

Department of Signals and Systems, Chalmers University of Technology, Gothenburg, Sweden

ABSTRACT

Received signal strength (RSS)-based single source localization when there is not a prior knowledge about the transmit power of the source is investigated. Because of nonconvex behavior of maximum likelihood (ML) estimator, convoluted computations are required to achieve its global minimum. Therefore, we propose a novel semidefinite programming (SDP) approach by approximating ML problem to a convex optimization problem which can be solved very efficiently. Computer simulations show that our proposed SDP has a remarkable performance very close to ML estimator. Linearizing RSS model, we also derive the partly novel least squares (LS) and weighted total least squares (WTLS) algorithms for this problem. Simulations illustrate that WTLS improves the performance of LS considerably.

Index Terms— Received signal strength (RSS), localization, semidefinite programming (SDP), weighted total least squares (WTLS), transmit power

1. INTRODUCTION

Wireless sensor network (WSN) has been emerged in many applications for monitoring, controlling, and tracking. The localization of a source sensor in a WSN is always the key problem. In a WSN localization problem, known-position sensors (anchor sensors) try to estimate the position of unknown-location sensors (source sensors) via noisy measurements [1]. Based on application requirements, accuracy, and efficiency, different types of measurements are employed in localization such as time-of-arrival, time-difference-of-arrival, received-signal-strength (RSS), and angle-of-arrival [1]. RSS is always an interesting method mainly because of its low complexity and cost of devices [1].

There are many localization techniques based on RSS measurements in the literature. The maximum likelihood (ML) estimator and the Cramér-Rao lower bound (CRLB) were derived in [2]. In addition, RSS linear estimators were studied in [3]. To compute ML solution, it is required to minimize a nonconvex cost function which is computationally intensive. Convergence problems of the ML estimator can be addressed by using semidefinite programming (SDP) techniques, in which the ML cost function is approximated with a convex function [4, 5, 6].

RSS-based localization requires a calibration between the source and anchors [1]. Since, in the RSS model, the measurement is a function of transmit power, finding the location of the source is not feasible as far as its transmit power is not available at anchors. Consequently, the source must transfer its transmit power to the anchors during RSS measurements which needs additional hardware in both source and anchors [1].

In this work, we assume that the anchors are not aware of the source transmit power. Dealing with this problem, in general, we introduce two methods. In the first method, we estimate the unknown transmit power along with source location (we call this method

URSS). In the second method, the dependency of the unknown transmit power in RSS model is eliminated from all measurements by using RSS difference between two anchors and a suitable estimator is applied in consequence; hereafter we call this method DRSS. For both methods, we propose a novel SDP approach to transform the ML or the nonlinear least squares (NLS) cost function to a convex one by using approximations and relaxation techniques. Our proposed SDP approach is different from those studied in [4, 5] since there is no information about the transmit power. We further linearize the measurement model and apply the least squares (LS) solution to the linear model. We also derive the novel weighted total least squares estimator to enhance the performance of LS [7, 8]. Although RSS localization algorithms are generally biased [2, 4], we employ the corresponding CRLB as a benchmark to compare the performance of proposed algorithms.

2. SYSTEM MODEL

Let $\mathbf{x}_s = [x_s, y_s]^T \in \mathbb{R}^2$ be the coordinates of the source to be determined. Denote by $\mathcal{C} = \{1, \dots, M\}$ the set of indices of the anchors connected to the source and by $\mathbf{x}_i = [x_i, y_i]^T \in \mathbb{R}^2$, $i \in \mathcal{C}$ the known location of anchor nodes. Under the log-distance path loss and log-normal shadowing model, the average received power (in dB) at i th anchor, P_i , is modeled as [2]

$$P_i = P_0 - 10\beta \log_{10} \frac{d_i}{d_0} + n_i, \quad i \in \mathcal{C}, \quad (1)$$

where P_0 is the reference power at reference distance d_0 (which depends on the transmit power), β is the path loss exponent, $d_i = \|\mathbf{x}_s - \mathbf{x}_i\|_2$ is the true distance between the source and the i th anchor, $\|\cdot\|_2$ denotes ℓ_2 norm, and n_i for $i \in \mathcal{C}$ are the log-normal shadowing term modeled as independent and identically distributed (iid) zero-mean Gaussian random variables with variance σ_{dB}^2 . Without loss of generality, it is assumed that $d_0 = 1$ m.

3. MAXIMUM LIKELIHOOD ESTIMATOR

Let $\boldsymbol{\theta} = [\mathbf{x}_s^T, P_0]^T$ be the unknown parameter vector to be estimated, the ML estimator based on the measurements in (1) is computed by following nonconvex optimization problem [9]

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \arg \min_{\boldsymbol{\theta}} \sum_{i \in \mathcal{C}} (P_i - P_0 + 10\beta \log_{10} d_i)^2. \quad (2)$$

We can express (2) alternatively as

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \arg \min_{\boldsymbol{\theta}} \sum_{i \in \mathcal{C}} \log_{10}^2 \frac{h_i \lambda_i}{\alpha}, \quad (3)$$

where $h_i \triangleq d_i^2$, $\lambda_i \triangleq 10^{P_i/5\beta}$, and $\alpha \triangleq 10^{P_0/5\beta}$. The solution of (3) is not closed-form, but can be approximated, for instance, by the Gauss-Newton (GN) method [9]. The drawback of the GN method is that it requires a good initialization to make sure that the algorithm converges to the global minimum [9].

4. SEMIDEFINITE PROGRAMMING

The cost function of the ML is severely nonlinear and nonconvex, therefore, finding its global minimum requires convoluted computations. By using SDP relaxation, we convert the ML problem to a convex optimization problem. The advantage of SDP problem over ML is that it can be solved with efficient computational methods that certainly converge to its global minimum [10].

As we mentioned earlier, we have two methods to deal with our problem. Let us start with the first method. Consider (1), by rearranging and diving both side by 5β , it can be reformulated as

$$\log_{10} d_i^2 \lambda_i = \frac{P_0}{5\beta} + \frac{n_i}{5\beta}, \quad i \in \mathcal{C}. \quad (4)$$

Taking power of 10 from both side yields

$$d_i^2 \lambda_i = \alpha 10^{n_i/5\beta}, \quad i \in \mathcal{C}. \quad (5)$$

For sufficiently small noise, the right hand side of (5) can be approximated using the first-order Taylor series expansion as

$$d_i^2 \lambda_i = \alpha \left(1 + \frac{\ln 10}{5\beta} n_i \right), \quad i \in \mathcal{C}. \quad (6)$$

This can be rewritten as

$$h_i \lambda_i = \alpha + n'_i, \quad i \in \mathcal{C}, \quad (7)$$

where n'_i is a zero-mean Gaussian random variable with variance $(\ln 10)^2 \alpha^2 \sigma_{dB}^2 / 25\beta^2$. Now, corresponding ML estimator of (7) is

$$\hat{\mathbf{x}}_s = \arg \min_{\mathbf{x}_s, \alpha} \sum_{i \in \mathcal{C}} (h_i \lambda_i - \alpha)^2. \quad (8)$$

To progress, we have to use another approximation. The ML estimator of (8) tries to minimize the ℓ_2 norm of the residual error. For sufficiently small residual error, we can approximate (8) by using ℓ_1 norm rather than ℓ_2 norm [10]

$$\hat{\mathbf{x}}_s = \arg \min_{\mathbf{x}_s, \alpha} \sum_{i \in \mathcal{C}} |h_i \lambda_i - \alpha|. \quad (9)$$

Indeed, we approximately convert the original ML cost function of (3) to another cost function (9). The cost function (9) is still nonlinear and nonconvex. In the next step, an auxiliary variable y is defined

$$h_i = d_i^2 = \|\mathbf{x}_s - \mathbf{x}_i\|^2 = y - 2\mathbf{x}_i^T \mathbf{x}_s + \mathbf{x}_i^T \mathbf{x}_i, \quad i \in \mathcal{C} \quad (10)$$

where $y = \mathbf{x}_s^T \mathbf{x}_s$. The minimization problem (9) can be relaxed to an SDP optimization problem as [10]

$$\min_{\mathbf{x}_s, \alpha, t_i, h_i, y} \sum_{i \in \mathcal{C}} t_i \quad (11a)$$

$$\text{s. t. } -t_i < h_i \lambda_i - \alpha < t_i, \quad (11b)$$

$$h_i = y - 2\mathbf{x}_i^T \mathbf{x}_s + \mathbf{x}_i^T \mathbf{x}_i, \quad (11c)$$

$$y \geq \mathbf{x}_s^T \mathbf{x}_s. \quad (11d)$$

Solution of (11) can be found effectively with optimal algorithms such as interior point method [10]. Moreover, convergence to the global minimum is guaranteed in SDP optimization problems [10].

Note, in (11), we have used the inequality constraint (11d) instead of the equality to relax our problem to a convex problem [10]. The inequality (11d) can be written as a linear matrix inequality (LMI) using the Schur complement [10]

$$\begin{bmatrix} y & \mathbf{x}_s^T \\ \mathbf{x}_s & \mathbf{I}_2 \end{bmatrix} \succeq \mathbf{0}_3. \quad (12)$$

Here, we continue with describing the SDP algorithm for the second method. We select an anchor as a reference (with index $r \in \mathcal{C}$) and calculate DRSS measurements. Hence (1) can be expressed as

$$P_{r,i} = P_r - P_i = 10\beta \log_{10} \frac{d_i}{d_r} + m_i, \quad i \in \mathcal{C}, i \neq r, \quad (13)$$

where P_r is the received power at the reference anchor, d_r is the distance between the reference anchor and the source, and $m_i = n_r - n_i$ is a zero-mean Gaussian random variable with variance $2\sigma_{dB}^2$. Since the noise of reference anchor appears in all DRSS measurements, they are correlated, which makes it difficult to relax the ML problem into an SDP problem. For this reason, we proceed with the NLS estimator instead. The NLS solution of (13) is [9]

$$\hat{\mathbf{x}}_s = \arg \min_{\mathbf{x}_s} \sum_{i \in \mathcal{C}, i \neq r} \left(P_{r,i} - 10\beta \log_{10} \frac{d_i}{d_r} \right)^2. \quad (14)$$

Using the procedure mentioned for previous case, we can approximate solution of (14) with the following optimization problem,

$$\hat{\mathbf{x}}_s = \arg \min_{\mathbf{x}_s} \sum_{i \in \mathcal{C}, i \neq r} |d_i^2 \vartheta_i - d_r^2|. \quad (15)$$

where $\vartheta_i \triangleq 10^{P_{r,i}/5\beta}$. The minimization problem (15) can be relaxed to an SDP optimization problem as [10]

$$\min_{\mathbf{x}_s, t_i, h_i, h_r, y} \sum_{i \in \mathcal{C}, i \neq r} t_i \quad (16a)$$

$$\text{s. t. } -t_i < h_i \vartheta_i - h_r < t_i, \quad (16b)$$

$$h_i = y - 2\mathbf{x}_i^T \mathbf{x}_s + \mathbf{x}_i^T \mathbf{x}_i, \quad (16c)$$

$$h_r = y - 2\mathbf{x}_r^T \mathbf{x}_s + \mathbf{x}_r^T \mathbf{x}_r, \quad (16d)$$

$$y \geq \mathbf{x}_s^T \mathbf{x}_s. \quad (16e)$$

Now, we have to pick up one of anchors as a reference. Note that the effect of log-normal shadowing is multiplicative to the distance in (1) [2], hence, long measured distances have higher error than short ones [2]. Consequently we select the nearest anchor to the source (the anchor with the highest RSS) as a reference anchor to prevent raising more errors in equations.

In summary, to apply the SDP solution for our localization problem, we have approximated the original cost function of ML (or NLS) to another cost function and then relaxed it to a convex problem. In the first step, we have substituted the function $\sum |\lambda_i h_i - \alpha|$ for the function $\sum \log_{10}^2(\lambda_i h_i / \alpha)$. Fig. 1a depicts two mentioned functions versus unknown parameters h and α (λ is a known parameter). To compare the cost functions of (3) and (9), we have used one realization. Five anchors are randomly placed in a square of 20×20 meters and a source located at $[10, 10]^T$. The standard deviation of the log-normal shadowing is 3 dB. Fig. 1b shows the cost function of the ML estimator given in (3) versus x and y coordinates when we have fixed the value of P_0 at the true value. It can be seen that the ML cost function has a global minimum at $[10.5, 11.5]^T$ (the step of mesh grid is 0.5) and some local minima and saddle points (e.g., a local minimum at $[2.5, 17.5]^T$). The cost function of (9) is shown in Fig. 1c which is much smoother than (3) and has a global minimum at $[10, 11.5]^T$. Fig. 1c still requires to be relaxed to a convex shape. In the next step, by using SDP relaxation of (11d), we transform function (9) to a convex function (11). Solution of (9) and (11) for source location will coincide, if the minimum of (11) occurs for $y = \mathbf{x}_s^T \mathbf{x}_s$ or if rank 1 condition for y is satisfied.

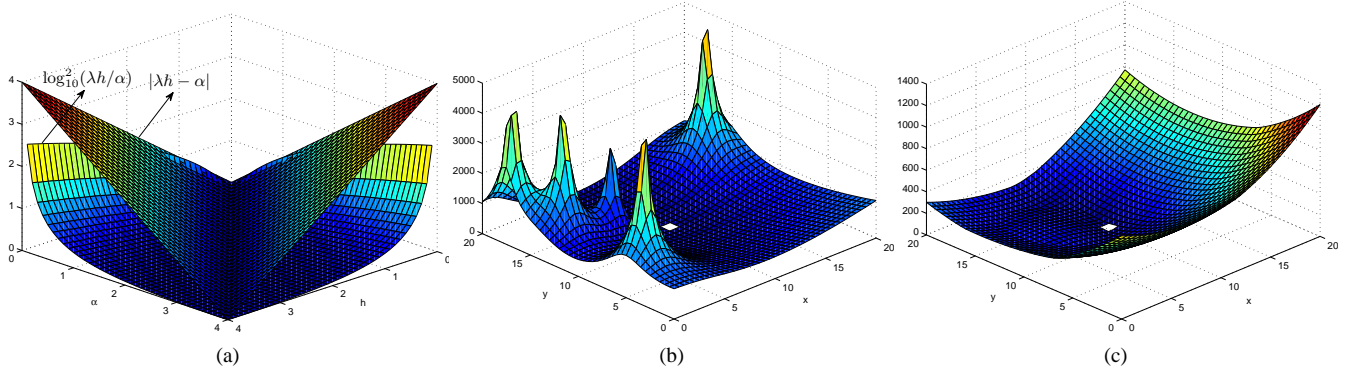


Fig. 1: (a) depiction of functions $|\lambda h - \alpha|$ and $\log_{10}^2(\lambda h/\alpha)$ versus unknown variables h and α (for simplicity, $\lambda = 1$). (b) cost function of (3), (c) cost function of (9) versus x and y coordinates, the minimum of the cost functions is indicated with white color.

5. LEAST SQUARES

In this section, we describe linear estimators for our localization model (1). Similar to the previous cases, we have two methods to deal with the unknown transmit power.

Consider (1), in the absence of noise, we can reformulate it as

$$d_i^2 = \zeta_i \alpha, \quad i \in \mathcal{C}, \quad (17)$$

where $\zeta_i \triangleq 10^{-P_i/5\beta}$. Let $\theta_1 = [\mathbf{x}_s^T, \mathbf{x}_s^T \mathbf{x}_s, \alpha]^T$ be the unknown vector to be estimated, and $k_i = \mathbf{x}_i^T \mathbf{x}_i$. Expanding and rearranging (17), we can express (17) in matrix form as

$$\mathbf{A}\theta_1 = \mathbf{b}, \quad (18)$$

where

$$\mathbf{A} = \begin{bmatrix} 2\mathbf{x}_1^T & -1 & \zeta_1 \\ \vdots & \vdots & \vdots \\ 2\mathbf{x}_M^T & -1 & \zeta_M \end{bmatrix}, \mathbf{b} = \begin{bmatrix} k_1 \\ \vdots \\ k_M \end{bmatrix}. \quad (19)$$

The LS solution of (18) is [9]

$$\hat{\theta}_{1,LS1} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}. \quad (20)$$

Now, we derive the LS estimator for the second method. Consider (13), we pick up an anchor as a reference and calculate the DRSS from the other anchors. In the absence of noise, (13) would be expressed as

$$d_i^2 \vartheta_i = d_r^2, \quad i \in \mathcal{C}, i \neq r, \quad (21)$$

Let $\theta_2 = [\mathbf{x}_s^T, \mathbf{x}_s^T \mathbf{x}_s]^T$ be the unknown vector to be estimated, then (21) can be expressed in matrix form as

$$\mathbf{P}\theta_2 = \mathbf{q}, \quad (22)$$

where

$$\mathbf{P} = \begin{bmatrix} \vdots & \vdots \\ (2\vartheta_i \mathbf{x}_i - 2\mathbf{x}_r)^T & (1 - \vartheta_i) \\ \vdots & \vdots \end{bmatrix}, \mathbf{q} = \begin{bmatrix} \vdots \\ \vartheta_i k_i - k_r \\ \vdots \end{bmatrix}, i \in \mathcal{C}, i \neq r. \quad (23)$$

The LS solution of (22) is [9]

$$\hat{\theta}_{2,LS2} = (\mathbf{P}^T \mathbf{P})^{-1} \mathbf{P}^T \mathbf{q}. \quad (24)$$

The reference anchor is selected as mentioned before for SDP-DRSS algorithm.

6. WEIGHTED TOTAL LEAST SQUARES

When we have measurement noise in the formulation of LS estimators, the disturbances appear in both data matrix and observation vector. However, LS only respects disturbances in the observation vector [9]. The more general case of LS is total least squares (TLS) which can tolerate disturbances in both data matrix and observation vector [11]. TLS assumes that errors in the data matrix and observation vector are equally sized, independent and identically distributed. However, this assumption is not valid in our expressions (18) and (22). The new approach, called weighted total least squares (WTLS), allows us to have unequally sized errors in both data matrix and observation vector [11]. The full details about finding the solution of a WTLS problem is given in [7, 8]. Briefly, the WTLS solution of (18) is obtained by the following optimization problem [7]

$$\hat{\theta}_{1,WTLS1} = \arg \min_{\theta_1} \sum_{i \in \mathcal{C}} \frac{r_i^2}{u_i}, \quad (25a)$$

$$\text{s.t. } r_i = \mathbf{a}_i^T \theta_1 - b_i, \quad (25b)$$

$$u_i = \theta_1^T \mathbf{W}_{11,i} \theta_1 - 2\theta_1^T \mathbf{W}_{12,i} + \mathbf{W}_{22,i}, \quad (25c)$$

where \mathbf{a}_i and b_i are the i th row of \mathbf{A} and the i th element of \mathbf{b} , respectively, and covariance matrices are

$$\mathbf{W}_{11,i} = \mathbb{E}[\mathbf{a}_i^T \mathbf{a}_i] = \text{Var}(\hat{\zeta}_i) \text{diag}[0, 0, 0, 1], \quad (26a)$$

$$\mathbf{W}_{12,i} = \mathbb{E}[b_i \mathbf{a}_i] = [0, 0, 0]^T, \quad (26b)$$

$$\mathbf{W}_{22,i} = \mathbb{E}[b_i^2] = 0, \quad i \in \mathcal{C}. \quad (26c)$$

Consider (18), the noise only appears in ζ_i . Let $\hat{\zeta}_i$ be the value of ζ_i corrupted by the noise given (1), then we have $\hat{\zeta}_i = \zeta_i 10^{n_i/5\beta}$. Since n_i is Gaussian random variable, $\hat{\zeta}_i$ has a log-normal distribution with variance

$$\text{Var}(\hat{\zeta}_i) = \hat{\zeta}_i^2 \left(e^{2\sigma_\zeta^2} + e^{\sigma_\zeta^2} \right), \sigma_\zeta = \frac{\sigma_{dB} \ln 10}{5\beta}, \quad i \in \mathcal{C}. \quad (27)$$

The cost function of WTLS (25) is nonlinear and has not any closed-form solution [7]. Solution of (25) can be obtained approximately by iterative optimization algorithms [7].

The corresponding WTLS solution of (22) can be derived in a similar manner.

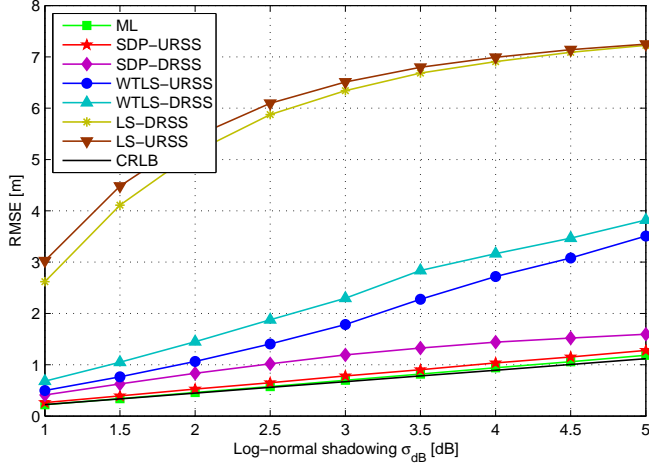


Fig. 2: The RMSE of the proposed algorithms.

7. SIMULATION RESULTS

In this section, we compare the performance of proposed algorithms through computer simulations. Twenty anchors were placed on the sides of a square of $20\text{ m} \times 20\text{ m}$ in equal distances and a source was randomly distributed inside the square. The values of P_0 and β were set to 30 and 4, respectively. Deriving the CRLB is straightforward and is not included here. The root mean square error (RMSE) of the proposed algorithms and CRLB were computed by averaging over all experiments. The cost function of ML and WTLSs were minimized by MATLAB routine `fminsearch`, with default setting, which uses Nelder-Mead Simplex method. The proposed SDP problems were solved by CVX toolbox [12].

The RMSE of the proposed algorithm versus the standard deviation of the log-normal shadowing is depicted in Fig. 2. The ML and WTLS algorithms were initialized with the true values to increase the probability of convergence to the global minimum. Fig. 2 shows that the performance of the LS algorithms are very poor since they do not respect errors appearing in the data matrix. As we expected, the WTLS estimators perform substantially better than LSs because they respect unequally sized disturbances in both the data matrix and observation vector. However, RMSE of WTLS-URSS is slightly lower than WTLS-DRSS. The reason is that in the derivation of WTLS, we assume that disturbances in each row of data matrix and observation vector are independent (row-wise WTLS [11]), but this assumption is not valid for WTLS-DRSS algorithm since the measurement noise of reference anchor appears in all rows and consequently the rows of data matrix and observation vector are correlated. Furthermore, Fig. 2 demonstrate that the ML has a superior performance to other algorithms and is only slightly worse than CRLB at low SNR. SDP-URSS performs remarkably, having a negligible gap with ML. SDP-DRSS performance is moderately worse than SDP-URSS because our SDP-DRSS does not handle the noise correlation due to the reference anchor. Fig. 3 depicts the cumulative density function (CDF) of the location error $\|\hat{\mathbf{x}}_s - \mathbf{x}_s\|_2$ of the proposed algorithms when the log-normal shadowing standard deviation is fixed at 3 dB. The order of the proposed algorithms is the same as in Fig. 2.

8. CONCLUSION

The single source RSS localization problem when the source transmit power is not available at anchors was treated in this paper. Two methods were introduced to deal with this problem: eliminating or

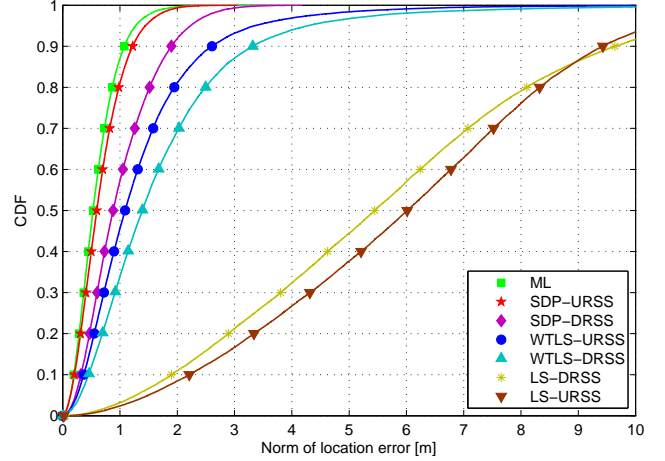


Fig. 3: The CDF of the proposed algorithms, $\sigma_{dB} = 3\text{ dB}$.

estimating the transmit power. A novel SDP approach by applying approximations and relaxations to ML (or NLS) estimator was derived. Although the ML estimator outperforms other algorithms, finding its global minimum involves complex computations and requires a good initialization. However, the proposed SDP approaches having an insignificant gap with ML performance can be solved efficiently without any initialization. Moreover, by linearizing the measurement equations, we derived the corresponding LS and WTLS estimators, and simulation results demonstrated that the WTLS has a notably better performance than LS.

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